Angular spectrum of quantized light beams

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We introduce a generalized angular spectrum representation for quantized light beams. By using our formalism, we are able to derive simple expressions for the electromagnetic vector potential operator in the case of: a) time-independent paraxial fields, b) time-dependent paraxial fields, and c) non-paraxial fields. For the first case, the well known paraxial results are fully recovered.

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Propagation of nonclassical states of the electromagnetic field is an issue of growing interest in quantum optics for both fundamental and technological purposes.[1] Consider, for instance, the relevance of propagation of entangled photons to quantum cryptographic systems.[2]

The purpose of the present Letter is to provide a novel perfectly general formalism for the representation of quantized light beams which can be used in *any* regime of propagation. Such objective is achieved by using a dispersion relation that some of us have recently introduced in Ref.,[3] and a generalized angular spectrum representation for the field operators.[4] The usefulness of our approach[5] becomes manifest whenever one deals with quantum systems for which both paraxial and non-paraxial regimes of propagation may be relevant as, e.g., down-converted photon pairs.[6]

Consider the plane-wave expansion of the positive-frequency part of the electromagnetic vector potential operator $\hat{\mathbf{A}}(\mathbf{r},t) = \hat{\mathbf{A}}^{(+)}(\mathbf{r},t) + \text{H.c.}$ in the Coulomb gauge[7]

$$\begin{aligned}\hat{\mathbf{A}}^{(+)}(\mathbf{r},t) &= \int \mathrm{d}^3\mathbf{k}\left(\frac{\hbar}{16\pi^3\varepsilon_0 c|\mathbf{k}|}\right)^{1/2} \\ &\quad\times \sum_{\lambda=1}^{2} \boldsymbol{\epsilon}^{(\lambda)}(\mathbf{k})\hat{a}_\lambda(\mathbf{k})\exp\left(\mathrm{i}\mathbf{k}\cdot\mathbf{r} - \mathrm{i}c|\mathbf{k}|t\right).\end{aligned} \tag{1}$$

Since we want to describe fields propagating mainly along the z axis, we find it convenient to define $k_z = s\zeta$, where $s \equiv \text{sign}(k_z) = \pm 1$, and $\zeta \geq 0$. Then we can rewrite Eq. (1) as

$$\begin{aligned}\hat{\mathbf{A}}^{(+)}(\mathbf{r},t) = &\sum_{s=\pm1}\int \mathrm{d}k_x \mathrm{d}k_y \int_0^\infty \mathrm{d}\zeta\left(\frac{\hbar}{16\pi^3\varepsilon_0 c|\mathbf{k}_s|}\right)^{1/2} \\ &\times \sum_{\lambda=1}^{2} \boldsymbol{\epsilon}^{(\lambda)}(\mathbf{k}_s)\hat{a}_\lambda(\mathbf{k}_s)\exp\left(\mathrm{i}\mathbf{k}_s\cdot\mathbf{r} - \mathrm{i}c|\mathbf{k}_s|t\right),\end{aligned} \tag{2}$$

where we have defined $\mathbf{k}_s = (k_x, k_y, s\zeta)$. The field annihilation and creation operators satisfy the canonical commutation relations[7]

$$[\hat{a}_\lambda(\mathbf{k}_s), \hat{a}^\dagger_{\lambda'}(\mathbf{k}'_{s'})] = \delta_{\lambda\lambda'}\delta_{ss'}\delta^{(2)}(\mathbf{q}-\mathbf{q}')\delta(\zeta-\zeta'). \tag{3}$$

Let $\omega \geq 0$ be an arbitrary frequency; at a later point in this Letter we shall identify ω with the carrier frequency of a paraxial field. We perform a change of variables $\{k_x, k_y, \zeta\} \to \{q_x, q_y, \omega\}$ such that

$$k_x = q_x, \quad k_y = q_y, \quad \zeta = f(\mathbf{q},\omega), \tag{4}$$

where $f(\mathbf{q},\omega) \geq 0$ is an (almost) arbitrary function to be determined. For reasons that will be soon clear, we require $f(\mathbf{q},\omega)$ to increase monotonically for increasing ω in the domain

$$\mathcal{I}_\omega(f,\mathbf{q}) = \left\{\omega \in \mathbb{R}^+ : f(\mathbf{q},\omega) \geq 0\right\}. \tag{5}$$

This condition implies that $\mathrm{d}f(\mathbf{q},\omega)/\mathrm{d}\omega > 0$ for $\omega \in \mathcal{I}_\omega(f,\mathbf{q})$. Therefore, in such domain, we can write

$$\begin{aligned}\delta^{(2)}(\mathbf{q}-\mathbf{q}')\delta(\zeta-\zeta') &= \delta^{(2)}(\mathbf{q}-\mathbf{q}')\delta\left[f(\mathbf{q},\omega) - f(\mathbf{q},\omega')\right] \\ &= \delta^{(2)}(\mathbf{q}-\mathbf{q}')\frac{\delta(\omega-\omega')}{\mathrm{d}f(\mathbf{q},\omega)/\mathrm{d}\omega}.\end{aligned} \tag{6}$$

If we substitute Eq. (6) into Eq. (3), we obtain

$$[\hat{a}_\lambda(\mathbf{k}_s), \hat{a}^\dagger_{\lambda'}(\mathbf{k}'_{s'})] = \delta_{\lambda\lambda'}\delta_{ss'}\delta^{(2)}(\mathbf{q}-\mathbf{q}')\frac{\delta(\omega-\omega')}{\mathrm{d}f(\mathbf{q},\omega)/\mathrm{d}\omega}. \tag{7}$$

Equation (7) suggests the introduction of the "angular-spectrum" field operators $\hat{a}_{\lambda s}(\mathbf{q},\omega)$ defined as[4]

$$\hat{a}_{\lambda s}(\mathbf{q},\omega) \equiv \hat{a}_\lambda(\mathbf{k}_s)\sqrt{\mathrm{d}f(\mathbf{q},\omega)/\mathrm{d}\omega}. \tag{8}$$

By using Eq. (8) and Eq. (7) it is easy to see that

$$[\hat{a}_{\lambda s}(\mathbf{q},\omega), \hat{a}^\dagger_{\lambda' s'}(\mathbf{q}',\omega')] = \delta_{\lambda\lambda'}\delta_{ss'}\delta^{(2)}(\mathbf{q}-\mathbf{q}')\delta(\omega-\omega'). \tag{9}$$

Equation (9) is the first main result of this Letter; it worthwhile to note that it is *exact*. No approximations were made to obtain it.

The condition $f(\mathbf{q},\omega) \geq 0$ defines a volume $\mathcal{V}_{\mathbf{q},\omega}(f)$ in the half-space $\mathbb{R}^2 \times \mathbb{R}^+$ spanned by $\{q_x, q_y, \omega\}$. This volume is bounded by the surface $\mathcal{S}_{\mathbf{q},\omega}(f) = \partial\mathcal{V}_{\mathbf{q},\omega}(f)$ defined by the equation $f(\mathbf{q},\omega) = 0$. If we define the 2-dimensional domain $\mathcal{C}_{\mathbf{q}}(f,\omega) = \left\{(q_x,q_y) \in \mathbb{R}^2 : f(\mathbf{q},\omega) \geq 0\right\}$ then, from Eq. (5) it readily follows

$$\int_{\mathbb{R}^2} \mathrm{d}^2\mathbf{q}\int_{\mathcal{I}_\omega(f,\mathbf{q})} \mathrm{d}\omega = \int_{\mathbb{R}^+} \mathrm{d}\omega \int_{\mathcal{C}_{\mathbf{q}}(f,\omega)} \mathrm{d}^2\mathbf{q}. \tag{10}$$

We use this equality to rewrite Eq. (2) immediately in the new variables $\{q_x, q_y, \omega\}$ as

$$\hat{\mathbf{A}}^{(+)}(\mathbf{r},t) = \sum_{s=\pm1}\int_0^\infty \mathrm{d}\omega \int_{\mathcal{C}_{\mathbf{q}}(f,\omega)} \mathrm{d}^2\mathbf{q}\left(\frac{\hbar/\sqrt{q^2+f^2}}{16\pi^3\varepsilon_0 c}\right)^{1/2} \times \sum_{\lambda=1}^{2}\boldsymbol{\epsilon}^{(\lambda)}(\mathbf{q},sf)\hat{a}_{\lambda s}(\mathbf{q},\omega)\sqrt{\mathrm{d}f/\mathrm{d}\omega} \times \exp\left(\mathrm{i}\mathbf{q}\cdot\mathbf{x} + \mathrm{i}sfz - \mathrm{i}tc\sqrt{q^2+f^2}\right), \tag{11}$$

where $q^2 = q_x^2 + q_y^2$, $f = f(\mathbf{q},\omega)$, and $\mathbf{x} = (x,y)$. Note that the square root of the Jacobian $J = \mathrm{d}f/\mathrm{d}\omega$ was used to pass from the original operators $\hat{a}_\lambda(\mathbf{k}_s)$ to the angular-spectrum operators $\hat{a}_{\lambda s}(\mathbf{q},\omega)$. Since we want to develop a formalism suitable for both non-paraxial and paraxial light beams, we rewrite Eq. (11) as

$$\hat{\mathbf{A}}^{(+)}(\mathbf{r},t) = \sum_{s=\pm1}\int_0^\infty \mathrm{d}\omega\, e^{-\mathrm{i}\omega(t-sz/c)}\,\hat{\mathbf{\Psi}}_s(\mathbf{r},t;\omega), \tag{12}$$

so that ω determines the plane carrier wave, and $\hat{\mathbf{\Psi}}_s(\mathbf{r},t;\omega)$ is the *envelope* field which, at this stage, is not required to be spatially and temporally slowly varying:

$$\begin{aligned}\hat{\mathbf{\Psi}}_s(\mathbf{r},t;\omega) \;=\; & \int_{\mathcal{C}_{\mathbf{q}}(f,\omega)} \mathrm{d}^2\mathbf{q}\left(\frac{\hbar\,\mathrm{d}f/\mathrm{d}\omega}{16\pi^3\varepsilon_0 c\sqrt{q^2+f^2}}\right)^{1/2} \\ & \times \sum_{\lambda=1}^{2}\boldsymbol{\epsilon}_s^{(\lambda)}(\mathbf{q},\omega)\hat{a}_{\lambda s}(\mathbf{q},\omega) \\ & \times \exp\left[\mathrm{i}\mathbf{q}\cdot\mathbf{x} + \mathrm{i}zs(f-\omega/c)\right] \\ & \times \exp\left[-\mathrm{i}t\left(c\sqrt{q^2+f^2}-\omega\right)\right].\end{aligned} \tag{13}$$

Moreover, since $\mathbf{k}_s = \hat{\mathbf{k}}_s\sqrt{q^2+f^2}$, where $\hat{\mathbf{k}}_s = (q\hat{\mathbf{q}} + sf\hat{\mathbf{z}})/\sqrt{q^2+f^2}$, and $\hat{\mathbf{q}} = \mathbf{q}/q$; we have defined $\boldsymbol{\epsilon}_s^{(\lambda)}(\mathbf{q},\omega) \equiv \boldsymbol{\epsilon}^{(\lambda)}(\mathbf{q},sf)$, where $\boldsymbol{\epsilon}_s^{(2)}(\mathbf{q},\omega) = s\hat{\mathbf{z}}\times\hat{\mathbf{q}}$, and

$$\boldsymbol{\epsilon}_s^{(1)}(\mathbf{q},\omega) = (f\hat{\mathbf{q}} - sq\hat{\mathbf{z}})/\sqrt{q^2+f^2}. \tag{14}$$

Until now, we furnished expressions for the field operators in the angular spectrum representation, but not for the energy, the momentum, etc. However, closed expressions for these physical quantities can be easily found by noting that the product

$$\hat{a}_\lambda^\dagger(\mathbf{k}_s)\hat{a}_\lambda(\mathbf{k}_s)\,\mathrm{d}\zeta = \hat{a}_{\lambda s}^\dagger(\mathbf{q},\omega)\hat{a}_{\lambda s}(\mathbf{q},\omega)\,\mathrm{d}\omega, \tag{15}$$

is invariant with respect to the change of variables Eq. (4). Then, for example, starting from the well known expression for the Hamiltonian operator of the electromagnetic field (see, e.g., Ref.[7]), after a straightforward calculation one obtain

$$\hat{H} = \frac{1}{2}\sum_{s=\pm1}\int_0^\infty \mathrm{d}\omega\int_{\mathcal{C}_{\mathbf{q}}(f,\omega)}\mathrm{d}^2\mathbf{q}\,\hbar c\sqrt{q^2+f^2(\mathbf{q},\omega)} \times \sum_{\lambda=1}^{2}\left[\hat{a}_{\lambda s}^\dagger(\mathbf{q},\omega)\hat{a}_{\lambda s}(\mathbf{q},\omega) + \hat{a}_{\lambda s}(\mathbf{q},\omega)\hat{a}_{\lambda s}^\dagger(\mathbf{q},\omega)\right]. \tag{16}$$

Similar calculations can be easily done for the other relevant quantities. Equation (16) shows that, as expected for an arbitrary field, the frequency ω of the carrier plane wave is not equal to the frequency $c|\mathbf{k}| = c\sqrt{q^2+f^2(\mathbf{q},\omega)}$ of the plane-wave mode $\exp(\mathrm{i}\mathbf{k}\cdot\mathbf{r})$. However, as we shall see later, $c|\mathbf{k}|$ reduces to ω in the paraxial limit.

At this point the function $f(\mathbf{q},\omega)$ is still undetermined, therefore we can exploit this freedom by imposing some constraints on the envelope field $\hat{\mathbf{\Psi}}_s(\mathbf{r},t;\omega)$ which is, until now, perfectly general. In particular, we want to find an expression for the envelope field in which the Fresnel propagator[6] plays a role even beyond the paraxial regime. To this end, we proceed as in Ref.[3] and we require $\hat{\mathbf{\Psi}}_s(\mathbf{r},t=0;\omega) \equiv \hat{\mathbf{\Psi}}_s(\mathbf{r};\omega)$ to satisfy the time-independent paraxial equation:

$$\frac{\partial^2\hat{\mathbf{\Psi}}_s(\mathbf{r};\omega)}{\partial x^2} + \frac{\partial^2\hat{\mathbf{\Psi}}_s(\mathbf{r};\omega)}{\partial y^2} + 2\mathrm{i}s\frac{\omega}{c}\frac{\partial\hat{\mathbf{\Psi}}_s(\mathbf{r};\omega)}{\partial z} = 0. \tag{17}$$

In this way we obtain an expression for $\hat{\mathbf{A}}^{(+)}(\mathbf{r},t)$ which is an *exact* solution of the full d'Alembert equation for any time $t > 0$ and its corresponding envelope field $\hat{\mathbf{\Psi}}_s(\mathbf{r},t=0;\omega)$ satisfies the time-independent paraxial wave equation at $t = 0$, as initial condition. If we substitute from Eq. (13) the plane wave $\exp\left[\mathrm{i}\mathbf{q}\cdot\mathbf{x} + \mathrm{i}zs(f-\omega/c)\right]$ into Eq. (17), we easily find

$$f(\mathbf{q},\omega) = \frac{\omega}{c}\left(1 - \frac{q^2c^2}{2\omega^2}\right). \tag{18}$$

For $\mathbf{q}\in\mathcal{C}_{\mathbf{q}}(f,\omega)$ this function satisfies all our requirements: it is positive and $\mathrm{d}f(\mathbf{q},\omega)/\mathrm{d}\omega = \left(1+\vartheta^2\right)/c > 0$ where we have defined[8] $\vartheta \equiv qc/(\sqrt{2}\omega)$. It is easy to see that the plane-wave frequency $c|\mathbf{k}| = \omega(1+\vartheta^4)^{1/2}$ reduces to ω in the paraxial limit $\vartheta \ll 1$. Finally, a closed expression for the field operator $\hat{\mathbf{A}}^{(+)}(\mathbf{r},t)$ can be given:

$$\begin{aligned}\hat{\mathbf{A}}^{(+)}(\mathbf{r},t) = & \sum_{s=\pm1}\int_0^\infty \mathrm{d}\omega\, e^{-\mathrm{i}\omega(t-sz/c)} \\ & \times \int_{\mathcal{C}_{\mathbf{q}}(f,\omega)}\mathrm{d}^2\mathbf{q}\left(\frac{\hbar(1+\vartheta^2)}{16\pi^3\varepsilon_0 c\omega\sqrt{1+\vartheta^4}}\right)^{1/2} \\ & \times \sum_{\lambda=1}^{2}\boldsymbol{\epsilon}_s^{(\lambda)}(\mathbf{q},\omega)\hat{a}_{\lambda s}(\mathbf{q},\omega)\exp\left(\mathrm{i}\mathbf{q}\cdot\mathbf{x} - \mathrm{i}s\frac{q^2c}{2\omega}z\right) \\ & \times \exp\left[-\mathrm{i}\omega t\left(\sqrt{1+\vartheta^4}-1\right)\right].\end{aligned} \tag{19}$$

Equation (19) is the second main result of this Letter. It is easy to recognize in the exponential function in the third row, the sought Fresnel propagator in momentum space. The spatial behavior of the envelope field is entirely governed by this term. It worth to note that Eq. (19) is *exact*, that is it has been obtained without any approximation and, therefore, it holds for both non-paraxial ($\vartheta \lesssim 1$) and paraxial ($\vartheta \ll 1$) beams. In the latter case, the slowly varying term $\exp\left[-\mathrm{i}\omega t\left(\sqrt{1+\vartheta^4}-1\right)\right]$ shows that the envelope field

$\hat{\boldsymbol{\Psi}}_s(\mathbf{r}, t; \omega)$ cannot be strictly monochromatic for any $t > 0$.

In the remaining part of this Letter, we give two different examples of the application of our theory in order to illustrate its generality. As a first example, let us generalize the previous case and require $\hat{\boldsymbol{\Psi}}_s(\mathbf{r}, t; \omega)$ to satisfy the *time-dependent* paraxial wave equation, for any t:[9]

$$\frac{\partial^2 \hat{\boldsymbol{\Psi}}_s}{\partial x^2} + \frac{\partial^2 \hat{\boldsymbol{\Psi}}_s}{\partial y^2} + 2\mathrm{i}s\frac{\omega}{c}\frac{\partial \hat{\boldsymbol{\Psi}}_s}{\partial z} + 2\mathrm{i}\frac{\omega}{c^2}\frac{\partial \hat{\boldsymbol{\Psi}}_s}{\partial t} = 0, \tag{20}$$

where $\hat{\boldsymbol{\Psi}}_s \equiv \hat{\boldsymbol{\Psi}}_s(\mathbf{r}, t; \omega)$ for short. If we substitute from Eq. (13) the relevant term $\exp[\mathrm{i}\mathbf{q}\cdot\mathbf{x} + \mathrm{i}zs(f - \omega/c)] \times \exp[-\mathrm{i}t(c\sqrt{q^2 + f^2} - \omega)]$ into Eq. (20), we obtain a new dispersion relation

$$f(\mathbf{q}, \omega) = \frac{\omega}{c}\left(1 - \frac{q^2 c^2}{4\omega^2}\right). \tag{21}$$

Once again, for $\mathbf{q} \in \mathcal{C}_{\mathbf{q}}(f, \omega)$ this function satisfies all our requirements: it is positive and $\mathrm{d}f(\mathbf{q}, \omega)/\mathrm{d}\omega = \left(1 + \eta^2\right)/c > 0$, where $\eta \equiv qc/(2\omega)$. It is easy to see that the plane-wave frequency $c|\mathbf{k}|$ becomes $c|\mathbf{k}| = \omega(1 + \eta^2)$. Also for this case a closed expression for the field operator $\hat{\mathbf{A}}^{(+)}(\mathbf{r}, t)$ can be given:

$$\begin{aligned} \hat{\mathbf{A}}^{(+)}(\mathbf{r}, t) = & \sum_{s=\pm 1} \int_0^\infty \mathrm{d}\omega\, e^{-\mathrm{i}\omega(t - sz/c)} \\ & \times \int_{\mathcal{C}_{\mathbf{q}}(f,\omega)} \mathrm{d}^2\mathbf{q} \left(\frac{\hbar}{16\pi^3 \varepsilon_0 c\omega}\right)^{1/2} \\ & \times \sum_{\lambda=1}^{2} \boldsymbol{\epsilon}_s^{(\lambda)}(\mathbf{q}, \omega)\hat{a}_{\lambda s}(\mathbf{q}, \omega) \exp\left[\mathrm{i}\mathbf{q}\cdot\mathbf{x} - \mathrm{i}\frac{q^2 c}{4\omega}(sz + ct)\right]. \end{aligned} \tag{22}$$

This expression is quite simpler than Eq. (19). However, its exponential part (in the last row), differ by a factor of 1/2 from the Fresnel propagator expression. Once again, we stress that Eq. (22) is exact, no approximation were made.

As a last example of application of our formalism, we choose to determine the function $f(\mathbf{q}, \omega)$ by requiring ω to coincide with the plane-wave frequency[4] $c|\mathbf{k}| : \omega = c|\mathbf{k}|$. It is easy to see that in this case we have

$$f(\mathbf{q}, \omega) = \frac{\omega}{c}\left(1 - \frac{q^2 c^2}{\omega^2}\right)^{1/2} \simeq \frac{\omega}{c}\left(1 - \frac{q^2 c^2}{2\omega^2}\right), \tag{23}$$

where the last approximate equality holds in the paraxial limit $qc/\omega \ll 1$. For $\mathbf{q} \in \mathcal{C}_{\mathbf{q}}(f, \omega)$ this function is positive and $\mathrm{d}f(\mathbf{q}, \omega)/\mathrm{d}\omega = 1/(c\sqrt{1 - (qc/\omega)^2}) > 0$, therefore all our requirements are fulfilled. As expected, in the paraxial limit Eq. (23) coincides with Eq. (18). Since by definition $\zeta = |k_z| = (\omega/c)|\cos\theta|$, it follows that $|\cos\theta| = f(\mathbf{q}, \omega)c/\omega$, and we can write

$$\begin{aligned} \hat{\mathbf{A}}^{(+)}(\mathbf{r}, t) = & \sum_{s=\pm 1} \int_0^\infty \mathrm{d}\omega\, e^{-\mathrm{i}\omega(t - sz/c)} \\ & \times \int_{\mathcal{C}_{\mathbf{q}}(f,\omega)} \mathrm{d}^2\mathbf{q} \left(\frac{\hbar/|\cos\theta|}{16\pi^3 \varepsilon_0 c\omega}\right)^{1/2} \sum_{\lambda=1}^{2} \boldsymbol{\epsilon}_s^{(\lambda)}(\mathbf{q}, \omega)\hat{a}_{\lambda s}(\mathbf{q}, \omega) \\ & \times \exp\left[\mathrm{i}\mathbf{q}\cdot\mathbf{x} - \mathrm{i}sz\omega(1 - |\cos\theta|)/c\right]. \end{aligned} \tag{24}$$

Equation (24) is our last result. It gives an exact expression for the electromagnetic potential vector operator of a generic, non-paraxial light beam, in the angular spectrum representation. By expanding in Taylor series the $|\cos\theta|$ term around $\theta = 0$, it is easy to see that Eq. (24) reduces to the well know classical paraxial expression (with the quantum operators $\hat{a}_{\lambda s}(\mathbf{q}, \omega)$ substituted by the corresponding classical amplitudes). Moreover, at the lowest order in θ, it coincides with Eq. (19) calculated at the lowest order in ϑ.

In conclusion, in this Letter we presented a novel formalism for the representation of arbitrary quantized light beams. First, we introduced an angular spectrum representation for the field annihilation and creation operators. Then, we used our formalism to derive an exact expression for the "paraxial-like" envelope field of a light beam. Finally, we illustrated the generality of our theory, by applying it to the description of time-dependent, paraxial and non-paraxial, light beams. It worth to note that, although our formalism is fully quantum, *all* the previous results can be straightforwardly extended to classical fields just by replacing the quantum operators $\hat{a}_{\lambda s}(\mathbf{q}, \omega)$ with the corresponding classical amplitudes $a_{\lambda s}(\mathbf{q}, \omega)$.

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